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On Cocommutative Hopf Algebras of Finite Representation Type

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Let $\mathscr{G}$ be a finite algebraic group, defined over an algebraically closed field k of characteristic $p>0$. Such a group decomposes into a semidirect product $\mathscr{G}=\mathscr{G}^0 \rtimes \mathscr{G}_{\mathrm{red}}$ with a constant group $\mathscr{G}_{\mathrm{red}}$ and a normal infinitesimal subgroup $\mathscr{G}^0$. If the principal block $\mathscr{B}_0(\mathscr{G})$ of the group algebra $H(\mathscr{G})$ has finite representation type, then both constituents have the same property, with at least one of them being semisimple. We determine the structure of the infinitesimal constituent $\mathscr{G}^0$ up to the classification of V-uniserial groups.

INTRODUCTION

This paper is concerned with the representation theory of finite-dimensional cocommutative Hopf algebras over algebraically closed fields of positive characteristic. As is well known, such an algebra can be viewed as the group algebra of a finite algebraic k-group $\mathscr{G}$. Special cases are the Hopf algebras associated to constant groups, i.e., the modular group algebras, as well as those of the infinitesimal groups of height $\leqslant 1$, that is, restricted enveloping algebras of restricted Lie algebras. The representation theory of both of these classes has received considerable attention. In either case one has fairly detailed information on the structure of the representation-finite and tame Hopf algebras (cf. [7, 9, 10, 16, 24, 25, 33]). Our ultimate goal will be the extension of these results to arbitrary cocommutative Hopf algebras.

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The main results of this paper, Theorems 2.7 and 3.1, provide a reduction to the classification of the so-called *V-uniserial* unipotent groups. The solution to this problem involves different techniques and is therefore relegated to a separate paper.

Our classification proceeds progressively, starting with infinitesimal groups of height $\leqslant 1$. From homological properties of restricted enveloping algebras one obtains the structure of restricted Lie algebras of finite representation type (cf. (1.3)), which turns out to be crucial for our approach to the general classification. In this context one main tool are rank varieties of modules, that were introduced by Carlson [5] for finite groups and Friedlander–Parshall [12] for infinitesimal groups of height $\leqslant 1$. Recent work by Friedlander–Suslin [14] and Suslin–Friedlander–Bendel [30, 31] affords the extension of the geometric techniques to arbitrary finite algebraic k-groups. The properties of the general varieties that are relevant for our purposes are summarized in Section 1.

In Section 2 we study infinitesimal groups of finite representation type. These are shown to be supersolvable, and thus possess a fairly tractable block structure (cf. (2.4)). Our main result here, Theorem 2.7, provides the structure of those infinitesimal groups, whose principal blocks are representation-finite. It turns out that these groups can be detected by investigating their second Frobenius kernels.

In Section 3 we address the general case by considering representations of finite algebraic groups. These are semidirect products of a constant and an infinitesimal part, allowing us to view the Hopf algebra as a skew group algebra over its infinitesimal component. It is shown that such a Hopf algebra is representation-finite if and only if both constituents have this property, with at least one of them being semisimple.

In retrospect it appears that the subject can be approached from two different points of view. The announcements [24, 33] delineate geometric methods that have led to precursors of our results. Homological techniques have proven to be more effective for the purposes of this paper, while geometric arguments will be an indispensible tool in our future work.

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1. PRELIMINARIES

Throughout we will be working over an algebraically closed field k of positive characteristic $p > 0$. Let $\mathscr{G}$ be a finite algebraic k-group with group algebra $H(\mathscr{G})$. In the sequel all $H(\mathscr{G})$-modules are assumed to be finite-dimensional left modules. For an $H(\mathcal{G})$-module M and a subalgebra $\Lambda \subset H(\mathcal{G})$, we will occasionally denote by $M|_\Lambda$ the restriction of M to Λ.

Given a $H(\mathcal{G})$-module M, Yoneda composition induces a natural homomorphism

$$\Phi_M : H^{ev}(\mathcal{G}, k) \to \operatorname{Ext}^\bullet_{H(\mathcal{G})}(M, M).$$

According to the proof of [14, (1.1)] this map endows the Yoneda algebra with the structure of a finitely generated $H^{ev}(\mathcal{G}, k)$-module. Following [31, Sect. 6] we define the *cohomological support variety* of M via

$$\mathcal{V}_{\mathcal{G}}(M) := Z(\ker \Phi_M) \subset \operatorname{maxspec}(H^{ev}(\mathcal{G}, k)).$$

Let Λ be an associative k-algebra, M a Λ-module with minimal projective resolution $(P_n)_{n\geqslant 0}$. Then

$$c_\Lambda(M) := \min\{c \in \mathbb{N}_0 \cup \{\infty\};\ \exists \lambda > 0 : \dim_k P_n \leqslant \lambda n^{c-1}\ \forall n \geqslant 1\}$$

is called the *complexity* of M. We say that M is *periodic*, if there exists $\ell > 0$ such that $P_{n+\ell} \cong P_n$ for $n \geqslant 1$. In particular, every projective module is periodic.

LEMMA 1.1. *Let $\mathcal{G}' \subset \mathcal{G}$ be a subgroup, M an $H(\mathcal{G})$-module. Then the following statements hold*:

(1) $\dim \mathcal{V}_{\mathcal{G}}(M) = c_{H(\mathcal{G})}(M)$.

(2) $\dim \mathcal{V}_{\mathcal{G}'}(M) \leqslant \dim \mathcal{V}_{\mathcal{G}}(M)$.

Proof. (1). Thanks to [14, (1.1)] the Yoneda algebra $\operatorname{Ext}^\bullet_{H(\mathcal{G})}(M, M)$ is a finitely generated $H^{ev}(\mathcal{G}, k)$-module, and the arguments of [3, (5.7.2)] yield the desired result.

(2) According to the Nichols–Zoeller Theorem [22, Theorem 7] (see also [23]), the algebra $H(\mathcal{G})$ is a free $H(\mathcal{G}')$-module. Consequently, a minimal projective resolution of M is also a projective resolution of $M|_{H(\mathcal{G}')}$, whence $c_{H(\mathcal{G}')}(M) \leqslant c_{H(\mathcal{G})}(M)$. ∎

We apply (1.1) to the case where $\mathcal{G}' = {}_{\mathcal{F}}\mathcal{G}$ is the first Frobenius kernel of $\mathcal{G}$. By general theory, the group algebra $H({}_{\mathcal{F}}\mathcal{G})$ is isomorphic to the restricted enveloping algebra $u(\operatorname{Lie}(\mathcal{G}))$ of the Lie algebra of $\mathcal{G}$. Owing to [11, (1.4)] the variety $\mathcal{V}_{{}_{\mathcal{F}}\mathcal{G}}(M)$ may be identified with

$$\mathcal{V}_{\operatorname{Lie}(\mathcal{G})}(M) := \{x \in \operatorname{Lie}(\mathcal{G});\ x^{[p]} = 0 \text{ and } M|_{\langle x\rangle} \text{ is not free}\} \cup \{0\}.$$

Here $\langle x \rangle$ denotes the subalgebra of $u(\operatorname{Lie}(\mathcal{G}))$ that is generated by x. This result has recently been generalized to infinitesimal groups by Friedlander, Suslin, and Bendel, and we refer the reader to [31] concerning the non-cohomological description of support varieties.

As an immediate application, we record the following basic criteria for blocks of finite and tame representation types.

THEOREM 1.2. *Let $\mathscr{B} \subset H(\mathscr{G})$ be a block, M a $\mathscr{B}$-module. Then the following statements hold*:

(1) *If $\mathscr{B}$ is representation-finite, then* $\dim \mathscr{V}_{\mathscr{G}}(M) \leqslant 1$.

(2) *If $\mathscr{B}$ is tame, then* $\dim \mathscr{V}_{\mathscr{G}}(M) \leqslant 2$.

Proof. According to a theorem of Larson and Sweedler [21], $H(\mathscr{G})$ is a Frobenius algebra. Consequently, the block $\mathscr{B} \subset H(\mathscr{G})$ is also Frobenius. Directly from the definition we obtain $c_{\mathscr{B}}(M) = c_{H(\mathscr{G})}(M)$.

If $\mathscr{B}$ has finite representation type, then every indecomposable $\mathscr{B}$-module is periodic (cf. [15]). Consequently, the complexity of every $\mathscr{B}$-module is $\leqslant 1$. In case $\mathscr{B}$ is tame, Rickard's Theorem [28, Theorem 2] implies $c_{\mathscr{B}}(M) \leqslant 2$. The assertions now follow directly from (1.1). ∎

Remark. Let $\mathscr{G}$ be an infinitesimal group of height $\leqslant 1$. According to [9, (3.2)] a block $\mathscr{B} \subset H(\mathscr{G})$ has finite representation type whenever there exists a simple $\mathscr{B}$-module with support variety of dimension $\leqslant 1$. By contrast, for $p = 2$ the group algebra of the quaternion group is tame with all modules having support varieties of dimension $\leqslant 1$.

The homological properties of restricted enveloping algebras afford the classification of those restricted Lie algebras $(L, [p])$, whose restricted enveloping algebras $u(L)$ are of finite representation type (cf. [9, (4.3)]). This result was first obtained by Pfautsch and Voigt [24, Theorem 3] and later re-discovered by Feldvoss and Strade [10, (2.4)].

Given a restricted Lie algebra $(L, [p])$, we denote by $N(L)$, and $T(L)$, the largest nilpotent, and toral ideal of L, respectively. By definition, the elements of $T(L)$ operate on L by semisimple transformations, so that $T(L)$ is contained in the center $C(L)$ of L. We say that L is *supersolvable* if its first derived algebra $[L, L]$ is nilpotent. If $V \subset L$ is a subspace, then $(V)_p$ denotes the p-subalgebra of L that is generated by V. An element $x \in L$ is referred to as *toral* (*p-nilpotent*) if $x^{[p]} = x$ ($x^{[p]^n} = 0$ for some $n \in \mathbb{N}$).

THEOREM 1.3. *The following statements are equivalent*:

(1) *$u(L)$ has finite representation type.*

(2) *There exist a toral element $t \in L$ and a p-nilpotent element $x \in L$ such that $L = kt \oplus N(L)$ and $N(L) = T(L) \oplus (kx)_p$.*

(3) $\dim \mathscr{V}_L(k) \leqslant 1$. ∎

2. INFINITESIMAL GROUPS OF FINITE MODULE TYPE

In this section we classify the infinitesimal groups whose principal blocks have finite representation type. We refer the reader to [6] concerning the terminology. Contrary to the notation used there, we will suppress the tilde for images and quotients in the category of finite algebraic groups. In addition, given a finite algebraic group $\mathcal{G}$, we will write $g \in \mathcal{G}$ instead of $g \in \mathcal{G}(R)$ for a commutative k-algebra R. The following result, which is crucial to our classification, illustrates the utility of the variety $\mathcal{V}_{\mathrm{Lie}(\mathcal{G})}(k)$ for infinitesimal k-groups.

Proposition 2.1. *Let $\mathcal{G}$ be an infinitesimal k-group such that* $\dim \mathcal{V}_{\mathrm{Lie}(\mathcal{G})}(k) \leqslant 1$. *Then $\mathcal{G}$ is supersolvable.*

Proof. We use induction on the height $h := h(\mathcal{G})$. If $h \leqslant 1$, then $H(\mathcal{G}) = u(\mathrm{Lie}(\mathcal{G}))$, and the assertion follows from (1.3). If $h > 1$, let ${}_{\mathcal{F}}\mathcal{G}$ be the first Frobenius kernel of $\mathcal{G}$. Then $\mathcal{H} := \mathcal{G}/{}_{\mathcal{F}}\mathcal{G}$ is a k-group of height $\leqslant h-1$, and the Frobenius homomorphism $\mathcal{F}: \mathcal{G} \to \mathcal{G}^{(p)}$ induces an injection $\mathcal{H} \hookrightarrow \mathcal{G}^{(p)}$ (cf. [6, II Sect. 7 (1.1), III Sect. 3 (3.2)]). In view of ([6, II Sect. 4 (1.4)]) we therefore have

$$\mathrm{Lie}(\mathcal{H}) \hookrightarrow \mathrm{Lie}(\mathcal{G}^{(p)}) = \mathrm{Lie}(\mathcal{G})^{(p)},$$

so that $\dim \mathcal{V}_{\mathrm{Lie}(\mathcal{H})}(k) \leqslant 1$. It thus follows from the inductive hypothesis that $\mathcal{H}$ is supersolvable.

We conclude the proof by showing that ${}_{\mathcal{F}}\mathcal{G}$ admits a $\mathcal{G}$-composition series with one-dimensional factors. Let $L := \mathrm{Lie}(\mathcal{G})$. Since $\dim \mathcal{V}_L(k) \leqslant 1$, we obtain from (1.3) that there exists a maximal torus $T \subset L' := L/T(L)$, and a p-nilpotent element $x \in L$ such that

$$L' = T \oplus (kx)_p$$

is the semidirect sum of T and the ideal $(kx)_p = \sum_{i=1}^{n} kx^{[p]^i}$. According to ([6, IV Sect. 3 (1.3)]) the ideal $T(L)$ is $\mathcal{G}$-invariant, so it suffices to find a $\mathcal{G}$-composition series of L'.

Since $[L' \otimes_k R, L' \otimes_k R]_p = [L', L']_p \otimes_k R$ for every commutative k-algebra R, the ideal $[L', L']_p$ is invariant under $\mathcal{G}$. As $[L', L']_p$ is abelian, the space generated by $([L', L']_p)^{[p^m]}$ is $\mathcal{G}$-invariant for every $m \geqslant 0$.

If L' is not abelian, then $(kx)_p = [L', L']_p$, so that this space possesses a $\mathcal{G}$-composition series with one-dimensional factors. As $\mathcal{G}$ is infinitesimal, the operation of $\mathcal{G}$ on the torus $L'/[L', L']_p \cong T$ is trivial, and the series can be continued to L'.

Alternatively, L' is abelian, and the p-powers of x define an appropriate $\mathcal{G}$-composition series for $L' = (kx)_p$.

We may now complete the proof by combining the series for $\mathscr{G}/_{\mathscr{F}}\mathscr{G}$ and $_{\mathscr{F}}\mathscr{G}$. ▮

Given a finite algebraic group $\mathscr{G}$, we will denote the principal block of $H(\mathscr{G})$ by $\mathscr{B}_0(\mathscr{G})$.

COROLLARY 2.2. *Let $\mathscr{G}$ be an infinitesimal k-group such that $\mathscr{B}_0(\mathscr{G})$ has finite representation type. Then $\mathscr{G}$ is supersolvable.*

Proof. According to (1.1) and (1.2) we have $\dim \mathscr{V}_{\mathrm{Lie}(\mathscr{G})}(k) \leqslant 1$, so that (2.1) yields the assertion. ▮

In order to obtain more definite information on the structure of representation-finite algebraic groups we require a few subsidiary results on supersolvable groups and their blocks. By rigidity (cf. [34, (7.7)]) any multiplicative normal subgroup of an infinitesimal group $\mathscr{G}$ is contained in the center $\mathscr{Z}(\mathscr{G})$ of $\mathscr{G}$. Consequently, the multiplicative constituent $\mathscr{M}(\mathscr{G})$ of $\mathscr{Z}(\mathscr{G})$ (cf. [34, (9.5)]) is the largest multiplicative normal subgroup of $\mathscr{G}$.

PROPOSITION 2.3. *Let $\mathscr{G}$ be an infinitesimal algebraic k-group. Then the following statements are equivalent*:

(1) *$\mathscr{G}$ is supersolvable*

(2) *$\mathscr{G}/\mathscr{M}(\mathscr{G}) \cong \mathscr{U} \rtimes \mathscr{M}$ is the semidirect product of a multiplicative group $\mathscr{M}$, and a unipotent normal subgroup $\mathscr{U}$.*

Proof. (1) $\Rightarrow$ (2). Without loss of generality we may assume that $\mathscr{M}(\mathscr{G}) = e_k$. Since $\mathscr{G}$ is supersolvable, there exists a composition series

$$\mathscr{G} =: \mathscr{G}_0 \supset \mathscr{G}_1 \supset \cdots \supset \mathscr{G}_n = e_k$$

of normal subgroups such that $\mathscr{G}_i/\mathscr{G}_{i+1}$ is isomorphic to ${}_p\alpha_k$ or ${}_p\mu_k$. In the latter case, [6, II Sect. 1 (2.11)] shows that $\mathscr{AUT}({}_p\mu_k) \cong \mathscr{AUT}((\mathbb{Z}/(p))_k)$. In view of ([6, II Sect. 1, No. 1, Remarque e]) this group is isomorphic to the constant group $(\mathrm{Aut}(\mathbb{Z}/(p)))_k$. Alternatively, we have $\mathscr{AUT}({}_p\alpha_k) \cong \mathscr{AUT}(\mathrm{Lie}({}_p\alpha_k)) \cong \mu_k$.

Conjugation gives rise to homomorphisms

$$\varrho_i\colon \mathscr{G} \to \mathscr{AUT}(\mathscr{G}_i/\mathscr{G}_{i+1}).$$

Since $\mathscr{M}(\mathscr{G}) = e_k$, it follows from [6, (IV Sect. 4 (1.11))] that the nilpotent normal subgroup

$$F(\mathscr{G}) := \bigcap_{i=0}^{n-1} \ker \varrho_i$$

is in fact unipotent.

Now let $\mathscr{H} \subset \mathscr{G}$ be a unipotent subgroup. By the above, the group $\varrho_i(\mathscr{H}) \subset \mathscr{AUT}(\mathscr{G}_i/\mathscr{G}_{i+1})$ is either constant or multiplicative. Since $\mathscr{H}$ is infinitesimal, we obtain $\varrho_i(\mathscr{H}) = e_k$ in the first case, while in the second case $\mathscr{H}$ being unipotent yields the same conclusion. It follows that

$$\mathscr{H} \subset F(\mathscr{G}),$$

so that $F(\mathscr{G})$ is the unique largest unipotent subgroup of $\mathscr{G}$. Consequently, $\mathscr{G}/F(\mathscr{G})$ contains no unipotent subgroups, and [6, IV Sect. 3 (3.7)] implies that this factor group is multiplicative. We may now apply [6, IV Sect. 2 (3.5)] to obtain the desired result.

(2) $\Rightarrow$ (1). Since $\mathscr{M}(\mathscr{G})$ is invariant, it suffices to show that $\mathscr{G}/\mathscr{M}(\mathscr{G})$ is supersolvable. This follows directly from the given structure. ∎

Let $\mathscr{G}$ be a finite algebraic k-group. The conjugation on $\mathscr{G}$ induces the (*left*) *adjoint representation* of $H(\mathscr{G})$ on itself. If η denotes the antipode of $H(\mathscr{G})$, then this action is given by

$$h \cdot x := \sum_{(h)} h_{(1)} x \, \eta(h_{(2)}) \qquad \forall h, x \in H(\mathscr{G}).$$

The set of algebra homomorphisms $H(\mathscr{G}) \to k$ coincides with the set $G(\mathcal{O}(\mathscr{G}))$ of group-like elements of the function algebra of $\mathscr{G}$, and thus has the structure of an abelian group via the convolution $*$.

Recall that $\mathscr{G}$ is *trigonalizable* if every simple $H(\mathscr{G})$-module has dimension 1. The following result generalizes [8, (4.4), (4.5)] to groups of arbitrary height.

PROPOSITION 2.4. *Let $\mathscr{G}$ be a supersolvable, infinitesimal k-group.*

(1) *If $\mathscr{G}$ is trigonalizable, then $H(\mathscr{G})$ possesses* $\dim_k H(\mathscr{M}(\mathscr{G}))$ *blocks, each of which has dimension* $\dim_k H(\mathscr{G}/\mathscr{M}(\mathscr{G}))$.

(2) *The canonical projection induces an isomorphism* $\mathscr{B}_0(\mathscr{G}) \cong H(\mathscr{G}/\mathscr{M}(\mathscr{G}))$.

Proof. (1) Since $H(\mathscr{G})$ is trigonalizable, the group $\mathscr{G}$ decomposes into a semidirect product

$$\mathscr{G} = \mathscr{U} \times \mathscr{M},$$

where $\mathscr{M} \supset \mathscr{M}(\mathscr{G})$ is multiplicative, and $\mathscr{U}$ is a unipotent normal subgroup of $\mathscr{G}$ (cf. [6, IV Sect. 2 (3.5)]. The function algebra $\mathcal{O}(\mathscr{M})$ of the multiplicative group $\mathscr{M}$ is isomorphic to the group algebra $k[G(\mathcal{O}(\mathscr{M}))]$. In the sequel we will occasionally identify $G(\mathcal{O}(\mathscr{M}))$ with the image of $G(\mathcal{O}(\mathscr{G}))$ under the canonical restriction map.

In view of the above decomposition, every simple $H(\mathcal{G})$-module is isomorphic to k_γ for a suitably chosen $\gamma \in G(\mathcal{O}(\mathcal{M})) =: \Gamma$.

We let $H(\mathcal{M})$ operate on $H(\mathcal{U})$ via the adjoint representation, and decompose

$$H(\mathcal{U}) = \bigoplus_{\psi \in R} H(\mathcal{U})_\psi$$

into its weight spaces. Thus, $R \subset \Gamma$, and we let $\Psi \subset \Gamma$ be the subgroup generated by R.

Given $\gamma \in \Gamma$, the module $P(\gamma) := H(\mathcal{G}) \otimes_{H(\mathcal{M})} k_\gamma$ is projective. The isomorphism

$$H(\mathcal{G}) \cong H(\mathcal{U}) \# H(\mathcal{M})$$

induces an isomorphism $H(\mathcal{G}) \cong H(\mathcal{U}) \otimes_k H(\mathcal{M})$ of $(H(\mathcal{U}), H(\mathcal{M}))$-bimodules. In particular, $P(\gamma)|_{H(\mathcal{U})} \cong H(\mathcal{U})$, so that $P(\gamma)$ is indecomposable. By the same token, we have

$$P(\gamma)|_{H(\mathcal{M})} \cong \bigoplus_{\psi \in \Psi} H(\mathcal{U})_\psi \otimes_k k_\gamma .$$

Let $\mathcal{B}_\gamma \subset H(\mathcal{G})$ be the block containing k_γ. By the above isomorphism, k_ζ belongs to $\mathcal{B}_\gamma$ if and only if $\zeta \in \Psi * \gamma$.

Consider $\mathcal{M}' := \operatorname{Spec}_k(k[\Gamma/\Psi]) \subset \mathcal{M}$. Since $\mathcal{M}(\mathcal{G})$ lies in the center of $\mathcal{G}$, we have $\mathcal{M}(\mathcal{G}) \subset \mathcal{M}'$. On the other hand, $\mathcal{M}'$ centralizes $H(\mathcal{U})$, and thus belongs to the center of $\mathcal{G}$. This implies $\mathcal{M}' \subset \mathcal{M}(\mathcal{G})$. It follows that $\dim_k H(\mathcal{M}(\mathcal{G})) = [\Gamma : \Psi]$. Consequently, $H(\mathcal{G})$ possesses $\dim_k H(\mathcal{M}(\mathcal{G}))$ blocks, and

$$\begin{aligned} \dim_k \mathcal{B}_\gamma &= \bigoplus_{\psi \in \Psi} \dim_k P(\psi * \gamma) = \dim_k H(\mathcal{U}) \operatorname{ord}(\Psi) \\ &= \dim_k H(\mathcal{U}) \frac{\operatorname{ord}(\Gamma)}{[\Gamma : \Psi]} = \dim_k H(\mathcal{G}/\mathcal{M}(\mathcal{G})). \end{aligned}$$

(2) Since $\mathcal{G}$ is supersolvable, the factor group $\mathcal{G}' := \mathcal{G}/\mathcal{M}(\mathcal{G})$ is trigonalizable with $\mathcal{M}(\mathcal{G}') = e_k$ (cf. (3.3)). According to (1) the algebra $H(\mathcal{G}')$ is connected. It follows that the restriction $\pi: \mathcal{B}_0(\mathcal{G}) \to H(\mathcal{G}')$ of the canonical projection maps the primitive central idempotent of $\mathcal{B}_0(\mathcal{G})$ onto the identity. Consequently, π is surjective. Since the ideal $H(\mathcal{G})\, H(\mathcal{M}(\mathcal{G}))^\dagger$ is generated by central idempotents not belonging to $\mathcal{B}_0(\mathcal{G})$, the map π is also injective, and our assertion follows. ∎

Remark. Let $\mathscr{G}$ be trigonalizable, $\mathscr{B}_\gamma \subset H(\mathscr{G})$ a block belonging to the simple module k_γ. It readily follows from the proof of (2.4) that the functor $M \mapsto M \otimes_k k_\gamma$ induces a Morita equivalence $\mathscr{B}_\gamma \sim \mathscr{B}_0(\mathscr{G})$. Since both blocks are basic, they are isomorphic (cf. [7, (I.2.6)]). Consequently, every block of $H(\mathscr{G})$ is isomorphic to $H(\mathscr{G}/\mathscr{M}(\mathscr{G}))$.

We continue by defining a class of infinitesimal groups that will play an important rôle in the classification of representation-finite cocommutative Hopf algebras. Given an infinitesimal group $\mathscr{G}$, we denote by $V_{\mathscr{G}}\colon \mathscr{G}^{(p)} \to \mathscr{G}$ the Verschiebung of $\mathscr{G}$ (cf. [6, IV Sect. 3 No. 4, II Sect. 7 No. 1]).

Definition. A commutative, unipotent infinitesimal k-group $\mathscr{U}$ is called *V-uniserial* if the Verschiebung induces an exact sequence

$$\mathscr{U}^{(p)} \xrightarrow{V_{\mathscr{U}}} \mathscr{U} \to {}_p\alpha_k \to e_k .$$

Let $\mathscr{U}$ be a commutative, unipotent infinitesimal group. Since $V_{\mathscr{U}}$ annihilates direct products of groups of type ${}_p\alpha_k$, its image is contained in the intersection of all maximal subgroups of $\mathscr{U}$. In case $\mathscr{U}$ is V-uniserial, $V_{\mathscr{U}}(\mathscr{U}^{(p)})$ is the unique maximal subgroup of $\mathscr{U}$. Thus, the images $V^i_{\mathscr{U}}(\mathscr{U}^{(p^i)})$ of the iterated Verschiebung are the only subgroups of $\mathscr{U}$. Consequently, $\mathscr{U}$ possesses exactly one composition series, i.e., $\mathscr{U}$ is uniserial.

Lemma 2.5. *Suppose that $p > 2$. Let $\mathscr{U}$ be a unipotent, infinitesimal k-group, $\mathscr{U}' \subset \mathscr{U}$ a central subgroup such that*

(a) $\mathscr{U}' \cong {}_p\alpha_k$, *and*
(b) $\mathscr{U}/\mathscr{U}'$ *is commutative of length* $\ell(\mathscr{U}/\mathscr{U}') = 2$, *and*
(c) $V_{\mathscr{U}/\mathscr{U}'} = 0$, *and*
(d) $\dim \mathscr{V}_{\mathscr{F}\mathscr{U}}(k) \leqslant 1$.

Then $\mathscr{U}$ contains a subgroup isomorphic to ${}_{p^2}\alpha_k$.

Proof. We put $\mathscr{W} := \mathscr{U}/\mathscr{U}'$. If $\mathscr{U}$ is commutative, then we consider the commutative diagram

$$\begin{tikzcd} \mathscr{U}^{(p)} \arrow[r, "V_{\mathscr{U}}"] \arrow[d, "\pi^{(p)}"] & \mathscr{U} \arrow[d, "\pi"] \\ \mathscr{W}^{(p)} \arrow[r, "V_{\mathscr{W}}"] & \mathscr{W} \end{tikzcd}$$

where the maps π and $\pi^{(p)}$ are the canonical projections. Since $V_{\mathscr{W}} = 0$, the Verschiebung $V_{\mathscr{U}}$ sends $\mathscr{U}^{(p)}$ into $\mathscr{U}'$. It now follows from (b) that $\ker V_{\mathscr{U}}$ has length $\geqslant 2$. As k is perfect, there exists a subgroup $\mathscr{K} \subset \mathscr{U}$ such that

$\ker V_{\mathcal{U}} = \mathcal{K}^{(p)}$. Owing to [6, IV Sect. 3 (6.11)] we thus have an isomorphism

$$\mathcal{K} \cong \prod_{i=1}^{s} {}_{p^{r_i}}\alpha_k .$$

From condition (d) we obtain

$$s = \dim \mathcal{V}_{\mathcal{F}\mathcal{K}}(k) \leqslant \dim \mathcal{V}_{\mathcal{F}\mathcal{U}}(k) \leqslant 1,$$

so that $r_1 = \ell(\mathcal{K}) \geqslant 2$.

In the general case we consider the commutator form

$$\chi\colon \mathcal{U} \times \mathcal{U} \to \mathcal{U}'; \qquad (a, b) \mapsto aba^{-1}b^{-1}.$$

Since $p > 2$, the map $\mathcal{U}' \to \mathcal{U}'$; $x \mapsto x^2$ is an automorphism, whose inverse we denote by $x \mapsto x^{1/2}$. We define a new composition on $\mathcal{U}$ via

$$a \circ b := ab\chi(a, b)^{-1/2} \qquad \forall a, b \in \mathcal{U}.$$

Direct computation reveals that this product endows $\mathcal{U}$ with the structure of a commutative group. Since the induced structures on $\mathcal{U}/\mathcal{U}'$ and $\mathcal{U}'$ coincide, $(\mathcal{U}, \circ)$ is, as an extension of two unipotent groups, unipotent (cf. [6, IV Sect. 2 (2.3.(c))]). By the same token, every subgroup of $(\mathcal{U}, \circ)$ is also a subgroup of $\mathcal{U}$. According to (d) $\mathcal{U}'$ is the only subgroup of $\mathcal{U}$ that is isomorphic to ${}_p\alpha_k$. Consequently, $(\mathcal{U}, \circ)$ has the same property, so that $(\mathcal{U}, \circ)$ also satisfies (d). By the first part of the proof, the commutative group $(\mathcal{U}, \circ)$ contains a subgroup $(\mathcal{K}, \circ) \cong {}_{p^2}\alpha_k$. Since the induced group structures on $\mathcal{U}/\mathcal{U}'$ and $\mathcal{U}'$ coincide, it follows that $(\mathcal{K}, \cdot)$ is a central extension of ${}_p\alpha_k$ by ${}_p\alpha_k$. Owing to [6, III Sect. 6 (7.7)] $\mathcal{K}$ is a commutative subgroup of $\mathcal{U}$, and the new structure coincides on $\mathcal{K}$ with the old one. ∎

Remark 3.8. Lemma (2.5) retains its validity for $p = 2$. In this case one verifies by a lengthy computation that the commutator form χ is the symmetrization of a suitably chosen bilinear form $\varphi\colon \mathcal{W} \times \mathcal{W} \to {}_2\alpha_k$, where $\mathcal{W} = {}_2\alpha_k \times {}_2\alpha_k$ or $\mathcal{W} = {}_{2^2}\alpha_k$. In the sequel we will take this fact, which will be published elsewhere, for granted.

Let $\mathcal{G}$ be an infinitesimal k-group with rth Frobenius kernel $\mathcal{F}_r\mathcal{G}$. According to [31, (5.2)] the dimension of $\mathcal{V}_{\mathcal{F}_r\mathcal{G}}(k)$ coincides with the dimension of the scheme $V_r(\mathcal{G})$ of homomorphisms from ${}_{p^r}\alpha_k$ to $\mathcal{G}$. This result will play a crucial rôle in the proof of the following proposition.

PROPOSITION 2.6. *Let $\mathcal{U}$ be a unipotent, infinitesimal k-group. Then the following statements are equivalent*:

(1) $\dim \mathscr{V}_{\mathscr{F}^2\mathscr{U}}(k) \leqslant 1$.

(2) *$\mathscr{U}$ is V-uniserial.*

(3) *$H(\mathscr{U})$ has finite representation type.*

Proof. (1) ⇒ (2) We proceed by induction on the length $\ell(\mathscr{U})$, the case $\ell(\mathscr{U})=1$ being trivial. Suppose that $\ell(\mathscr{U}) \geqslant 2$. According to [6, IV Sect. 4 (1.3)] the nilpotent group $\mathscr{U}$ has a non-trivial center, so that its Lie algebra contains a one-dimensional unipotent restricted subalgebra. Consequently, there exists a central subgroup $\mathscr{U}' \subset \mathscr{U}$ that is isomorphic to ${}_p\alpha_k$.

Consider $\mathscr{W} := \mathscr{U}/\mathscr{U}'$. If $\dim \mathscr{V}_{\mathscr{F}\mathscr{W}}(k) \geqslant 2$, then $\mathscr{U}$ contains a subgroup $\tilde{\mathscr{U}}$ such that $\tilde{\mathscr{U}}/\mathscr{U}' \cong {}_p\alpha_k \times {}_p\alpha_k$. We may now apply (2.5) to see that $\tilde{\mathscr{U}}$ also contains a subgroup that is isomorphic to ${}_{p^2}\alpha_k$. This, however, contradicts (1).

We therefore have $\dim \mathscr{V}_{\mathscr{F}\mathscr{W}}(k) \leqslant 1$. Suppose that $\dim \mathscr{V}_{\mathscr{F}^2\mathscr{W}}(k) \geqslant 2$. Since $\dim \mathscr{V}_{\mathscr{F}\mathscr{W}}(k) \leqslant 1$ [31, (5.2)] implies the existence of a subgroup $\tilde{\mathscr{U}} \subset \mathscr{U}$ such that $\tilde{\mathscr{U}}/\mathscr{U}' \cong {}_{p^2}\alpha_k$. Consequently, Lemma 2.5 yields again a contradiction.

By what we have just shown, we have $\dim \mathscr{V}_{\mathscr{F}^2\mathscr{W}}(k) \leqslant 1$, and the inductive hypothesis implies that $\mathscr{W}$ is V-uniserial.

Since $\mathscr{W}$ is abelian, the commutator $[a, b] := aba^{-1}b^{-1}$ belongs to $\mathscr{U}'$ for every $a, b \in \mathscr{U}$. As $\mathscr{U}'$ lies centrally in $\mathscr{U}$ there results an alternating bilinear map

$$\chi\colon \mathscr{W} \times \mathscr{W} \to \mathscr{U}'; \qquad (\bar{a}, \bar{b}) \mapsto [a, b].$$

From [6, IV Sect. 3 (4.7)] we obtain the identity

$$V_{\mathscr{U}'}(\chi^{(p)}(\mathscr{F}(\bar{a}), \bar{b})) = \chi(\bar{a}, V_{\mathscr{W}}(\bar{b})).$$

Since $V_{\mathscr{U}'}\colon \mathscr{U}'^{(p)} \to \mathscr{U}'$ is trivial, the inverse image $\mathscr{Z}$ of $V_{\mathscr{W}}(\mathscr{W}^{(p)})$ under the canonical projection $\mathscr{U} \to \mathscr{W}$ is a central subgroup of length $\ell(\mathscr{U})-1$. It follows that

$$H(\mathscr{U}/\mathscr{Z}) \cong H({}_p\alpha_k) \cong k[x]/(x^p).$$

Consequently, $H(\mathscr{Z})$ is central in $H(\mathscr{U})$, and letting $X \in H(\mathscr{U})$ be a preimage of x under $H(\mathscr{U}) \to H(\mathscr{U}/\mathscr{Z})$, we obtain

$$H(\mathscr{U}) = H(\mathscr{Z})[X] + H(\mathscr{Z})^{\dagger}\, H(\mathscr{U}).$$

Since the augmentation ideal $H(\mathscr{Z})^{\dagger}$ is the radical of $H(\mathscr{Z})$, this implies that $H(\mathscr{U}) = H(\mathscr{Z})[X]$ is abelian. As a result, the group $\mathscr{U}$ is commutative.

In order to show that $\mathcal{U}$ is V-uniserial, we consider a subgroup $\mathcal{K} \subset \mathcal{U}$ such that $\mathcal{K}^{(p)} = \ker V_{\mathcal{U}}$. Owing to [6, IV Sect. 3 (6.11)] we have

$$\mathcal{K} \cong \prod_{i=1}^{s} {}_{p^{r_i}}\alpha_k .$$

From (1.1) we obtain

$$s = \dim \mathscr{V}_{\mathcal{F}\mathcal{K}}(k) \leqslant \dim \mathscr{V}_{\mathcal{F}^2\mathcal{K}}(k) \leqslant \dim \mathscr{V}_{\mathcal{F}^2\mathcal{U}}(k) \leqslant 1,$$

so that $\mathcal{K} \cong {}_{p^r}\alpha_k$. If $r \geqslant 2$, then ${}_{\mathcal{F}^2}\mathcal{K} \cong {}_{p^2}\alpha_k$, and $2 = c_{H({}_{\mathcal{F}^2}\mathcal{K})}(k) = \dim \mathscr{V}_{\mathcal{F}^2\mathcal{K}}(k)$, a contradiction. Hence the kernel of $V_{\mathcal{U}}$ has length 1 and $\mathcal{U}$ is V-uniserial.

(2) $\Rightarrow$ (3) Let $H(\mathcal{U})^{\dagger}$ denote the augmentation ideal of $H(\mathcal{U})$. Since $H(\mathcal{U})$ is commutative, $H(\mathcal{U}) = \mathcal{O}(\mathcal{D}(\mathcal{U}))$ is the function algebra of the Cartier dual $\mathcal{D}(\mathcal{U})$ of $\mathcal{U}$, and we thus have

$$\dim_k \operatorname{Lie}(\mathcal{D}(\mathcal{U})) = \dim_k H(\mathcal{U})^{\dagger}/(H(\mathcal{U})^{\dagger})^2.$$

As $\mathcal{U}$ is V-uniserial [6, IV Sect. 3 (4.9)] provides an exact sequence

$$e_k \to {}_p\alpha_k \to \mathcal{D}(\mathcal{U}) \xrightarrow{\mathcal{F}} \mathcal{D}(\mathcal{U}^{(p)}),$$

so that $\dim_k \operatorname{Lie}(\mathcal{D}(\mathcal{U})) = \dim_k \operatorname{Lie}({}_{\mathcal{F}}\mathcal{D}(\mathcal{U})) = 1$. By the above identity this shows that the local algebra $H(\mathcal{U})$ is a truncated polynomial ring in one variable, and thus of finite representation type.

(3) $\Rightarrow$ (1) This follows directly from (1.1) and (1.2). ∎

Let Λ be a Frobenius algebra with Nakayama automorphism $\mu: \Lambda \to \Lambda$. Given a minimal projective resolution $(P_n, \partial_n)_{n \geqslant 0}$ of a Λ-module M, we define $\Omega^n(M) := \ker \partial_{n-1}$ for $n \geqslant 1$. The resulting operator Ω is called the *loop space operator* or *Heller operator* (cf. [1, 2] for details). Closely related is the so-called *Auslander–Reiten translation* τ. In our context, we have

$$\tau(M) = \Omega^2(M^{(1)}),$$

where $M^{(1)}$ is obtained from M by twisting the action by μ^{-1}. In particular, $\tau(M)$ is simple if and only if $\Omega^2(M)$ has this property. We will require this fact in the proof of the following result.

Recall that Λ is a *Nakayama algebra* if and only if each principal indecomposable Λ-module is uniserial. According to [9, (3.2)] the representation-finite blocks of restricted enveloping algebras are Nakayama algebras.

THEOREM 2.7. *Let $\mathcal{G}$ be an infinitesimal k-group. Then the following statements are equivalent:*

(1) *$\mathcal{B}_0(\mathcal{G})$ has finite representation type.*

(2) *$\mathcal{G}/\mathcal{M}(\mathcal{G}) \cong \mathcal{U} \rtimes {}_{p^n}\mu_k$ is a semidirect product with a V-uniserial normal subgroup $\mathcal{U}$.*

(3) *$H(\mathcal{G})$ is a Nakayama algebra.*

(4) $\dim \mathcal{V}_{\mathcal{F}^2\mathcal{G}}(k) \leqslant 1$.

(5) *$H({}_{\mathcal{F}^2}\mathcal{G})$ is a Nakayama algebra.*

Proof. (1) $\Rightarrow$ (2) Let $\mathcal{G}' := \mathcal{G}/\mathcal{M}(\mathcal{G})$. Since $\mathcal{B}_0(\mathcal{G})$ has finite representation type, we may apply (2.2) and (2.4) consecutively to see that $H(\mathcal{G}')$ has the same property. As $\mathcal{M}(\mathcal{G}') = e_k$, (2.3) provides a decomposition

$$\mathcal{G}' = \mathcal{U} \rtimes \mathcal{M},$$

with a multiplicative subgroup $\mathcal{M}$, and a unipotent normal subgroup $\mathcal{U}$.

Since $H(\mathcal{G}')$ is representation-finite, (1.1) and (1.2) imply

$$\dim \mathcal{V}_{\mathcal{F}^2\mathcal{U}}(k) \leqslant \dim \mathcal{V}_{\mathcal{U}}(k) \leqslant \dim \mathcal{V}_{\mathcal{G}'}(k) \leqslant 1.$$

Owing to (2.6) this shows that $\mathcal{U}$ is V-uniserial.

In order to see that $\mathcal{M} \cong {}_{p^n}\mu_k$, we let $H(\mathcal{M})$ act on $H(\mathcal{U})$ via the adjoint representation. Since $H(\mathcal{M})$ is semisimple, there results a weight space decomposition

$$H(\mathcal{U}) = \bigoplus_{\alpha \in R} H(\mathcal{U})_\alpha,$$

where R is a finite subset $G(\mathcal{O}(\mathcal{M}))$. Note that $H(\mathcal{U})_\alpha\, H(\mathcal{U})_\beta \subset H(\mathcal{U})_{\alpha * \beta}$. Moreover, since $\mathcal{U}$ is V-uniserial, we have $\dim_k H(\mathcal{U})^\dagger/(H(\mathcal{U})^\dagger)^2 = 1$, so that there exists a root vector $x \in H(\mathcal{U})_\alpha$ such that $\{1, x, x^2, ..., x^{p^r-1}\}$ is a basis of $H(\mathcal{U})$. Thus,

$$R = \{\alpha^i; 0 \leqslant i \leqslant p^r - 1\}.$$

Consider $\alpha: \mathcal{M} \to \mu_k$, the character given by $\alpha \in G(\mathcal{O}(\mathcal{M}))$. By the above observation, $\ker \alpha$ operates trivially on $H(\mathcal{U})$. Thus, $\ker \alpha \subset \mathcal{M}(\mathcal{G}') = e_k$, proving that α is injective.

If the height of $\mathcal{M}$ equals n, then $\dim_k \mathcal{O}(\mathcal{M}) \geqslant p^n$ and $\operatorname{im} \alpha \subset {}_{p^n}\mu_k$. It follows from [6, I Sect. 5 (1.5)] that $\mathcal{O}(\alpha): \mathcal{O}({}_{p^n}\mu_k) \to \mathcal{O}(\mathcal{M})$ is surjective, implying that $\mathcal{M}$ and ${}_{p^n}\mu_k$ have the same order. Hence $\mathcal{O}(\alpha)$ is bijective, and α has the same property.

(2) $\Rightarrow$ (3) By assumption, the algebra $H(\mathscr{U}) \cong k[X]/(X^{p^m})$ is a Nakayama algebra. We have seen in the proof of (2.4) that $P(\lambda) := H(\mathscr{G}') \otimes_{H({}_{p^n}\mu_k)} k_\lambda$ is the projective cover of the simple $H(\mathscr{G}')$-module k_λ. By the same token, $P(\lambda)|_{H(\mathscr{U})} \cong H(\mathscr{U})$ is a uniserial $H(\mathscr{U})$-module. Since the powers of the augmentation ideal $H(\mathscr{U})^\dagger$ are $\mathscr{G}'$-invariant, it follows that every $H(\mathscr{U})$-submodule of $P(\lambda)$ is a $H(\mathscr{G}')$-submodule. Consequently, $P(\lambda)$ is a uniserial $H(\mathscr{G}')$-module. As every simple $H(\mathscr{G}')$-module is one-dimensional, it follows that $H(\mathscr{G}')$ is a Nakayama algebra.

We continue by showing that any block $\mathscr{B}(\mathscr{G}) \subset H(\mathscr{G})$ containing a one-dimensional module is a Nakayama algebra. According to [32, (2.37)] every simple $\mathscr{B}(\mathscr{G})$ module is of the form k_λ for some algebra homomorphism $\lambda: H(\mathscr{G}) \to k$. Let M be a $H(\mathscr{G})$-module such that $H(\mathscr{M}(\mathscr{G}))$ operates trivially on M. Since $H(\mathscr{M}(\mathscr{G}))$ is semisimple, the canonical projection $H(\mathscr{G}) \to H(\mathscr{G}')$ is easily seen to induce an isomorphism

$$H^1(\mathscr{G}', M) \cong H^1(\mathscr{G}, M).$$

Given simple $\mathscr{B}(\mathscr{G})$-modules k_λ, k_μ, we have

$$\operatorname{Ext}^1_{H(\mathscr{G})}(k_\lambda, k_\mu) \cong H^1(\mathscr{G}, \operatorname{Hom}_k(k_\lambda, k_\mu)) \cong H^1(\mathscr{G}, k_{\mu * \lambda^{-1}}).$$

Also note that since the modules belong to the same block, the central algebra $H(\mathscr{M}(\mathscr{G}))$-operates via the same morphism, i.e., $\lambda|_{H(\mathscr{M}(\mathscr{G}))} = \mu|_{H(\mathscr{M}(\mathscr{G}))}$. Hence $\mu * \lambda^{-1}$ factors through to a morphism $H(\mathscr{G}') \to k$, which we will also denote by $\mu * \lambda^{-1}$.

Let k_λ be a simple $\mathscr{B}(\mathscr{G})$-module. Then we have, observing [18, Theorem 9],

$$\sum_\mu \dim_k \operatorname{Ext}^1_{H(\mathscr{G})}(k_\lambda, k_\mu) = \sum_\mu \dim_k H^1(\mathscr{G}', k_{\mu * \lambda^{-1}}) \leqslant 1.$$

Here the sum extends over those μ for which k_μ is a simple $\mathscr{B}(\mathscr{G})$-module. Another application of [18, Theorem 9] now shows that $\mathscr{B}(\mathscr{G})$ is a Nakayama algebra.

Now let $\mathscr{B}(S) \subset H(\mathscr{G})$ be an arbitrary block, belonging to the simple module S. Without loss of generality we may assume S to be non-projective. If T is another simple $\mathscr{B}(S)$-module, then $\mathscr{M}(\mathscr{G})$ operates via the same character on S and T, and thereby trivially on $\operatorname{Hom}_k(S, T)$. Since $\mathscr{G}/\mathscr{M}(\mathscr{G})$ is trigonalizable, $\operatorname{Hom}_k(S, T)$ contains a one-dimensional $\mathscr{G}$-module. It follows that there exists λ such that $T \cong S \otimes_k k_\lambda$. Since the block corresponding to k_λ is a Nakayama algebra, our introductory remarks in conjunction with [1, (IV.2.10)] yield $\Omega^2(k_\lambda) \cong k_\mu$, for a suitable algebra homomorphism μ. General principles (cf. [2, (3.1.6)]) now provide isomorphisms

$$\Omega^2(S \otimes_k k_\lambda) \oplus (\text{proj}) \cong S \otimes_k \Omega^2(k_\lambda) \cong S \otimes_k k_\mu .$$

Since $S \otimes_k k_\mu$ is simple and not projective, we obtain that $\Omega^2(T) \cong S \otimes_k k_\mu$ is simple, and we may apply [1, (IV.2.10)] to see that $\mathscr{B}(S)$ is a Nakayama algebra.

(3) $\Rightarrow$ (4) According to [1, (VI.2.1)] $H(\mathscr{G})$ has finite representation type, so that (1.1) and (1.2) yield

$$\dim \mathscr{V}_{\mathscr{F}^2\mathscr{G}}(k) \leqslant \dim \mathscr{V}_{\mathscr{G}}(k) \leqslant 1.$$

(4) $\Rightarrow$ (5) We put $\tilde{\mathscr{G}} := {}_{\mathscr{F}^2}\mathscr{G}$ and $\mathscr{H} := \tilde{\mathscr{G}}/\mathscr{M}(\tilde{\mathscr{G}})$. A consecutive application of (2.1) and (2.4) shows that $\mathscr{B}_0(\tilde{\mathscr{G}}) \cong H(\mathscr{H})$, so that we also have $\dim \mathscr{V}_{\mathscr{H}}(k) \leqslant 1$. According to (2.6) the unipotent part of $\mathscr{H}$ is V-uniserial, and the arguments of (2) $\Rightarrow$ (3) now show that (5) holds.

(5) $\Rightarrow$ (1) We apply [1, (VI.2.1)] and (1.2) to see that

$$\dim \mathscr{V}_{\mathscr{F}^2\mathscr{G}}(k) \leqslant 1.$$

Now (1.1) and (2.1) ensure the supersolvability of $\mathscr{G}$.

We put $\mathscr{G}' := \mathscr{G}/\mathscr{M}(\mathscr{G})$, and propose to show that $\dim \mathscr{V}_{\mathscr{F}^2\mathscr{G}'}(k) \leqslant 1$. Suppose first that ${}_p\alpha_k \subset \mathscr{G}'$, and let $\mathscr{E}$ be the inverse image of ${}_p\alpha_k$ under the natural projection $\pi: \mathscr{G} \to \mathscr{G}'$. There results an exact sequence

$$e_k \to \mathscr{M}(\mathscr{G}) \to \mathscr{E} \to {}_p\alpha_k \to e_k\,.$$

Owing to [6, IV Sect. 3 (3.7)] the group $\mathscr{E} \subset \mathscr{G}$ contains a subgroup isomorphic to ${}_p\alpha_k$.

Now let ${}_{p^2}\alpha_k \subset \mathscr{G}'$. We claim that $\mathscr{G}$ contains a subgroup isomorphic to ${}_{p^2}\alpha_k$. Using induction on the length of the kernel, we shall verify this claim for any surjection $\pi: \mathscr{G} \to \mathscr{G}'$ with multiplicative kernel $\mathscr{Q}$. We assume that $\ell(\mathscr{Q}) \geqslant 1$. According to [6, IV Sect. 1 (1.1, 1.2)] there exists a subgroup $\mathscr{T} \subset \mathscr{Q}$ such that $\mathscr{Q}/\mathscr{T} \cong {}_p\mu_k$. As before, we let $\mathscr{E}$ be the inverse image of ${}_{p^2}\alpha_k$ under the natural projection π. We put $\mathscr{E}_1 := \mathscr{E}/\mathscr{T}$ and consider the exact sequence

$$e_k \to {}_p\mu_k \to \mathscr{E}_1 \xrightarrow{\pi_1} {}_{p^2}\alpha_k \to e_k\,.$$

By composing the commutator product with the natural embedding ${}_p\mu_k \hookrightarrow \mu_k$ we obtain a commutator form

$$\chi: {}_{p^2}\alpha_k \times {}_{p^2}\alpha_k \to \mu_k; \qquad \chi(\pi_1(x), \pi_1(y)) = xyx^{-1}y^{-1}.$$

According to [6, I Sect. 2 (7.1)] the form χ induces a homomorphism $\psi: {}_{p^2}\alpha_k \to \mathscr{D}({}_{p^2}\alpha_k)$ into the Cartier dual of ${}_{p^2}\alpha_k$. Since $\mathscr{O}(\mathscr{D}({}_{p^2}\alpha_k)) \cong H({}_{p^2}\alpha_k) \cong k[X, Y]/(X^p, Y^p)$ the latter group has height $\leqslant 1$, implying that the kernel of ψ is non-trivial. It readily follows that the center $\mathscr{Z}_1$ of $\mathscr{E}_1$ has length $\geqslant 2$.

Since $\mathscr{Z}_1/{}_p\mu_k$ embeds into ${}_{p^2}\alpha_k$, $\mathscr{Z}_1$ is not multiplicative. Thanks to [6, IV Sect. 3 (3.7)] $\mathscr{Z}_1$ thus contains a copy of ${}_p\alpha_k$. The corresponding subgroup $\mathscr{N} \subset \mathscr{Z}_1$ is normal in $\mathscr{E}_1$, and $\mathscr{E}_2 := \mathscr{E}_1/\mathscr{N}$ has length 2. Since $\mathscr{E}_2$ is not multiplicative, the above arguments imply that $\mathscr{E}_2 \cong {}_p\mu_k \times {}_p\alpha_k$. Let $\mathscr{K}$ be the inverse image of ${}_p\alpha_k$ under the canonical projection $\mathscr{E}_1 \to \mathscr{E}_2$. Thanks to [6, IV Sect. 2 (2.3(c))] and [6, III Sect. 6 (7.7)] the group $\mathscr{K}$ is unipotent and commutative of length 2. Since $\mathscr{K}$ intersects ${}_p\mu_k$ trivially, the map π_1 provides an embedding $\mathscr{K} \hookrightarrow {}_{p^2}\alpha_k$. As both groups have length 2, it follows that $\mathscr{K} \cong {}_{p^2}\alpha_k$.

Our arguments show the existence of a subgroup of $\mathscr{E}_1 \subset \mathscr{G}/\mathscr{T}$ that is isomorphic to ${}_{p^2}\alpha_k$. By inductive hypothesis that group possesses a lifting in $\mathscr{G}$.

As an upshot of our discussion, we see that the projection π induces a surjection $V_2(\mathscr{G})(k) \to V_2(\mathscr{G}')(k)$ between the varieties of homomorphisms ${}_{p^2}\alpha_k \to \mathscr{G}$ and ${}_{p^2}\alpha_k \to \mathscr{G}'$. Since $\dim V_2(\mathscr{G}) = \dim V_2(\mathscr{G})(k)$ (cf. [6, I Sect. 3 (6.9)]) an application of [31, (5.2)] shows

$$\dim \mathscr{V}_{\mathscr{F}^2\mathscr{G}'}(k) \leqslant \dim \mathscr{V}_{\mathscr{F}^2\mathscr{G}}(k) \leqslant 1.$$

It now follows from (2.6) that the unipotent part of $\mathscr{G}'$ is V-uniserial. Accordingly, the arguments of (2) $\Rightarrow$ (3) show that $\mathscr{B}_0(\mathscr{G})$ has finite representation type. ∎

3. REPRESENTATION-FINITE GROUPS OF DIMENSION ZERO

In this section we conclude the reduction by studying the structure of cocommutative Hopf algebras of finite representation type. As before we assume the scheme theoretic point of view and consider $H = H(\mathscr{G})$, where $\mathscr{G} = \operatorname{Spec}_k(A)$ is a finite algebraic k-group. It is well known (cf. [6, II Sect. 5 (2.4)]) that such a group decomposes into a semidirect product

$$\mathscr{G} = \mathscr{G}^0 \rtimes \mathscr{G}_{\text{red}},$$

where $\mathscr{G}_{\text{red}}$ is a constant group, and $\mathscr{G}^0$ is an infinitesimal, normal subgroup. Consequently,

$$H(\mathscr{G}) \cong H(\mathscr{G}^0) \,\#\, H(\mathscr{G}_{\text{red}}) \cong H(\mathscr{G}^0)[\mathscr{G}_{\text{red}}]$$

is a skew group algebra.

THEOREM 3.1. *Let $\mathcal{G}$ be a finite algebraic k-group. Then the following statements are equivalent:*

(1) *$\mathcal{B}_0(\mathcal{G})$ has finite representation type.*

(2) (a) *$H(\mathcal{G}^0)$ and $H(\mathcal{G}_{\mathrm{red}})$ are of finite representation type.*

(b) *$H(\mathcal{G}^0)$ or $H(\mathcal{G}_{\mathrm{red}})$ is semisimple.*

(3) *$H(\mathcal{G})$ has finite representation type.*

Proof. (1) $\Rightarrow$ (2)(a) Let $\mathcal{N} \subset \mathcal{G}$ be a normal subgroup, and consider the canonical projection π: $H(\mathcal{G}) \to H(\mathcal{G}/\mathcal{N})$. Since the image of $\mathcal{B}_0(\mathcal{G})$ is a two-sided ideal of $H(\mathcal{G}/\mathcal{N})$ that does not annihilate the trivial module, it follows that $\mathcal{B}_0(\mathcal{G}/\mathcal{N})$ is a direct summand of $\pi(\mathcal{B}_0(\mathcal{G}))$. In particular, $\mathcal{B}_0(\mathcal{G}/\mathcal{N})$ has finite representation type.

By setting $\mathcal{N} := \mathcal{G}^0$, we obtain that $\mathcal{B}_0(\mathcal{G}_{\mathrm{red}})$ has finite representation type, and [2, (6.3.5)] implies that the defect group of $\mathcal{B}_0(\mathcal{G}_{\mathrm{red}})$ is cyclic. Since this group is a Sylow-p-subgroup of $\mathcal{G}_{\mathrm{red}}$, we may apply Higman's Theorem [16, Theorem 4] to see that $H(\mathcal{G}_{\mathrm{red}})$ is representation-finite.

We next prove that $H(\mathcal{G}^0)$ has finite representation type. Since $\mathcal{B}_0(\mathcal{G})$ enjoys this property, a consecutive application of (1.1) and (1.2) yields

$$\dim \mathcal{V}_{\mathcal{F}^2\mathcal{G}^0}(k) \leqslant \dim \mathcal{V}_{\mathcal{G}}(k) \leqslant 1.$$

We may now apply (2.7) to see that $H(\mathcal{G}^0)$ has finite representation type.

(b) We first observe that $\mathcal{G}_{\mathrm{red}}$ leaves the variety $\mathcal{V}_{\mathrm{Lie}(\mathcal{G})}(k)$ invariant. As $\mathcal{B}_0(\mathcal{G})$ has finite representation type, (1.1) and (1.2) yield

$$\dim \mathcal{V}_{\mathrm{Lie}(\mathcal{G})}(k) \leqslant \dim \mathcal{V}_{\mathcal{G}}(k) \leqslant 1.$$

Since the projective variety $\mathrm{Proj}(\mathcal{V}_{\mathrm{Lie}(\mathcal{G})}(k))$ is connected (cf. [11, (2.2)]), it follows that there is $x \in \mathrm{Lie}(\mathcal{G})$ such that $\mathcal{V}_{\mathrm{Lie}(\mathcal{G})}(k) = kx$. If $x = 0$, then [6, IV Sect. 3 (3.7)] implies the semisimplicity of $H(\mathcal{G}^0)$. Alternatively, let $g \in \mathcal{G}_{\mathrm{red}}$ be an element of order p. Since the automorphism group of kx is the multiplicative group $k^\times$, it follows that the cyclic group $C_p := \langle g \rangle$ operates trivially on kx. As a result, the subalgebra $H(C_p \times {}_p\alpha_k)$ is isomorphic to

$$H(C_p) \otimes_k H({}_p\alpha_k) \cong H(C_p) \otimes_k u(kx) \cong k[X, Y]/(X^p, Y^p).$$

Observing (1.1) we obtain

$$2 = c_{H(C_p \times {}_p\alpha_k)}(k) = \dim \mathcal{V}_{C_p \times {}_p\alpha_k}(k) \leqslant \dim \mathcal{V}_{\mathcal{G}}(k),$$

a contradiction. Consequently, $\mathcal{G}_{\mathrm{red}}$ does not possess elements of order p. Hence $p \nmid \mathrm{ord}(\mathcal{G}_{\mathrm{red}})$, and Maschke's theorem shows that $H(\mathcal{G}_{\mathrm{red}})$ is semisimple.

(2) $\Rightarrow$ (3) If the group algebra $H(\mathcal{G}_{\mathrm{red}})$ is semisimple, then the extension $H(\mathcal{G})\colon H(\mathcal{G}^0)$ is separable, and it follows from (a) and [19, Theorem 4] that $H(\mathcal{G})$ has finite representation type.

Now suppose $H(\mathcal{G}^0)$ to be semisimple. Since $H(\mathcal{G}) \cong H(\mathcal{G}^0)[\mathcal{G}_{\mathrm{red}}]$ is a skew group algebra with coefficients in a semisimple commutative k-algebra, direct computation shows that the extension $H(\mathcal{G})\colon H(\mathcal{G}_{\mathrm{red}})$ is separable. Thus, [19, Theorem 4] implies that $H(\mathcal{G})$ is representation-finite.

(3) $\Rightarrow$ (1) This is trivial. ∎

Remark. Suppose that $H(\mathcal{G})$ has finite representation type. According to [1, (IV.2.14)], (2.7), and (3.1) $H(\mathcal{G})$ is a Nakayama algebra in case $H(\mathcal{G}_{\mathrm{red}})$ is semisimple.

4. REPRESENTATION-FINITE STABILIZERS

In this final section we illustrate how Theorem 1.3 is related to recent work by A. Premet [27]. In order to avoid case-by-case considerations, we assume throughout that $p > 5$. Given a restricted Lie algebra $(L, [p])$, and a linear form $\chi \in L^*$, we consider the following factor algebra of the universal enveloping algebra $\mathcal{U}(L)$ of L:

$$u(L, \chi) := \mathcal{U}(L)/(\{x^p - x^{[p]} - \chi(x)^p\, 1;\, x \in L\}).$$

The *reduced enveloping algebra* $u(L, \chi)$ is a finite-dimensional Frobenius algebra, which is usually not even an augmented algebra. If $\chi = 0$, then $u(L) = u(L, 0)$ is the restricted enveloping algebra of L. We refer the reader to [29] for basic properties of reduced enveloping algebras.

Let $\mathcal{G}$ be a connected reductive group with Lie algebra L. The almost simple constituents of the derived group $(\mathcal{G}, \mathcal{G})$ of $\mathcal{G}$ will be denoted $\mathcal{G}_1, \ldots, \mathcal{G}_s$. Then there exists a torus $T_0 \subset L$ such that

$$L = T_0 \oplus \sum_{i=1}^{s} \mathrm{Lie}(\mathcal{G}_i).$$

It follows that a p-nilpotent element $e \in L$ is contained in $\sum_{i=1}^{s} \mathrm{Lie}(\mathcal{G}_i)$. Given an element $x \in L$, we denote its Jordan decomposition (cf. [29, (II.3.5)]) by

$$x = x_s + x_n,$$

and write $C_L(x)$ for the centralizer of $x \in L$.

LEMMA 4.1. *Let $x \in L$ be an element such that $u(C_L(x))$ has finite representation type. Then either $C_L(x_s)$ is a torus, or there exists a torus $T \subset L$ such that $C_L(x_s) \cong s\ell(2) \oplus T$ is a direct sum of Lie algebras.*

Proof. Let $\mathscr{H} := C_{\mathscr{G}}(x_s)^0 \subset \mathscr{G}$ be the connected stabilizer of $x_s \in L$ relative to the adjoint representation. According to general theory, $\mathscr{H}$ is reductive with Lie algebra $C_L(x_s)$ (see for instance [4, (IV.13.19), (III.9.1)]). Since $C_L(x) = C_{\mathrm{Lie}(\mathscr{H})}(x_n)$, we may assume that $x =: e$ is p-nilpotent. We put $L_i := \mathrm{Lie}(\mathscr{G}_i)$ $1 \leqslant i \leqslant s$ and write $e = \sum_{i=1}^{s} e_i$, with $e_i \in L_i$.

Let $i \in \{1, ..., s\}$. In virtue of [4, (14.25)] there exists a Borel subalgebra $B_i \subset L_i$ such that $e_i \in B_i$. We recall that the unipotent radical $V_i = \bigoplus_{n \in \mathbb{N}} V_i^n$ of B_i is $\mathbb{Z}$-graded via the heights of the positive roots. Since the nilpotent element e_i lies in V_i, we obtain $(\mathrm{ad}\, e_i)(V_i) \subset \sum_{n \geqslant 2} V_i^n$, whence

$$\dim_k C_{V_i}(e) = \dim_k C_{V_i}(e_i) \geqslant \dim_k V_i^1 =: r_i, \tag{1}$$

where r_i is the rank of $\mathscr{G}_i$. As $u(C_L(e))$ is representation-finite, the support variety $\mathscr{V}_{C_L(e)}(k)$ has dimension $\leqslant 1$. Since the ideals L_i commute pairwise, it follows that at most one of the varieties $\mathscr{V}_{C_{L_i}(e)}(k)$ is one-dimensional. Thus, without loss of generality, we have

$$\dim \mathscr{V}_{C_{L_i}(e)}(k) \leqslant \delta_{i1} \qquad 1 \leqslant i \leqslant s.$$

Consequently, $C_{L_i}(e)$ is a torus for $i \geqslant 2$. In view of (1) we conclude that $s \leqslant 1$.

If $s = 0$, then L is a torus and $e = 0$. We therefore assume that $s = 1$ and set $r := r_1$, $V := V_1$.

Since $\dim \mathscr{V}_{C_V(e)}(k) = 1$, (1.3) provides a p-nilpotent element x such that $C_V(e) = (kx)_p$. We have $(kx)_p = \bigoplus_{i=0}^{n-1} kx^{[p]^i}$ and $x^{[p]^n} = 0$. Thus, (1) implies $x^{[p]^{r-1}} \neq 0$.

We write $V^{(i)} := \sum_{n \geqslant i} V^n$. Since the grading of V is compatible with the p-map, we have $x^{[p]^{r-1}} \in V^{(p^{r-1})}$. Let h be the Coxeter number of the group $\mathscr{G}$. Then $h - 1$ is the height of the maximal root of the root system of $\mathscr{G}$ (cf. [17, p. 84]), and $V^{(p^{r-1})} \neq (0)$ yields

$$p^{r-1} \leqslant h - 1.$$

As $p > 5$, the formulae for h (cf. [17, p. 80]) entail $r = 1$. Thus $L_1 \cong s\ell(2)$ is a complete ideal of L, whose factor algebra L/L_1 is a torus. This readily implies our assertion. ∎

We are now in a position to retrieve, for $p > 5$, a recent result of Premet (cf. [27, (5.3)]). Let $\mathscr{G}$ be a reductive group, whose derived group $(\mathscr{G}, \mathscr{G})$

is simply connected. Following [20] we decompose a linear form $\chi \in L^*$ into its semisimple and nilpotent parts:

$$\chi = \chi_s + \chi_n .$$

We denote by $\mathscr{G}^\chi$ and L^χ the stabilizers of χ in $\mathscr{G}$ and L, respectively.

THEOREM 4.2 (A. Premet). *Let $\chi \in L^*$ be a linear form. Then the following statements are equivalent:*

(1) *$u(L, \chi)$ has finite representation type*

(2) *Either L^{χ_s} is a torus, or there exists a torus $T \subset L$ such that $L^{\chi_s} \cong s\ell(2) \oplus T$ is a direct sum of Lie algebras, and $\chi_n([L^{\chi_s}, L^{\chi_s}]) \neq (0)$.*

Proof. (1) ⇒ (2) By applying Premet's theorem [27, (2.4)] and (1.3) consecutively, we see that $u(L^\chi)$ has finite representation type. According to [26, (3.1)] the group $\mathscr{H} := \mathscr{G}^{\chi_s}$ is reductive with Lie algebra L^{χ_s}, and there exists a p-nilpotent element $e \in L^{\chi_s}$ such that $C_{L^{\chi_s}}(e) \subset L^\chi$. Thus, $u(C_{L^{\chi_s}}(e))$ has finite representation type, and (4.1) gives the desired structure of L^{χ_s}.

We now invoke [13, (3.2)], according to which the algebras $u(L, \chi)$ and $u(L^{\chi_s}, \chi_n|_{L^{\chi_s}})$ are Morita equivalent. Since

$$u(L^{\chi_s}, \chi_n|_{L^{\chi_s}}) \cong \bigoplus_{i=1}^{\dim_k T} u(s\ell(2), \chi_n|_{s\ell(2)})$$

has finite representation type, it follows that χ_n does not vanish on $s\ell(2) = [L^{\chi_s}, L^{\chi_s}]$.

(2) ⇒ (1) This follows directly from the above Morita equivalence. ∎

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